Math. Z. (2012) 270:759–781
DOI 10.1007/s00209-010-0825-z

Mathematische Zeitschrift

Higman ideal, stable Hochschild homology and Auslander-Reiten conjecture

Yuming Liu · Guodong Zhou · Alexander Zimmermann

Received: 7 May 2010 / Accepted: 30 November 2010 / Published online: 15 January 2011

Abstract Let A and B be two finite dimensional algebras over an algebraically closed field, related to each other by a stable equivalence of Morita type. We prove that A and B have the same number of isomorphism classes of simple modules if and only if their 0-degree Hochschild Homology groups $HH_0(A)$ and $HH_0(B)$ have the same dimension. The first of these two equivalent conditions is claimed by the Auslander-Reiten conjecture. For symmetric algebras we will show that the Auslander-Reiten conjecture is equivalent to other dimension equalities, involving the centers and the projective centers of A and B. This motivates our detailed study of the projective center, which now appears to contain the main obstruction to proving the Auslander-Reiten conjecture for symmetric algebras. As a by-product, we get several new invariants of stable equivalences of Morita type.

The first author was supported by Marie Curie Fellowship IIF. The second and the third authors were supported by a German-French grant "PROCOPE" of the DAAD, respectively a French-German grant "partenariat Hubert Curien PROCOPE dossier 14765WB". The second author benefits also from financial support via postdoctoral fellowship from the network "Representation theory of algebras and algebraic Lie theory" and the DAAD.

Y. Liu
Laboratory of Mathematics and Complex Systems, School of Mathematical Sciences,
Beijing Normal University, Beijing 100875, People's Republic of China
e-mail: ymliu@bnu.edu.cn

G. Zhou (✉)
Ecole Polytechnique Fédérale de Lausanne, SB/IGAT/CTG,
Batîment MA B3 424, 1015 Lausanne, Switzerland
e-mail: guodong.zhou@epfl.ch

A. Zimmermann
Département de Mathématiques et LAMFA (UMR 6140 du CNRS),
Université de Picardie, 33 rue St Leu, 80039 Amiens Cedex 1, France
e-mail: alexander.zimmermann@u-picardie.fr

Keywords Auslander-Reiten conjecture · Higman ideal · Projective center · Stable equivalence of Morita type · Stable Hochschild homology · Transfer map

Mathematics Subject Classification (2010) 16G10 · 16E40

1 Introduction

Equivalences between stable module categories occur in many places in representation theory and algebra and are closed related to derived equivalences. A result by Rickard [32] and Keller-Vossieck [12] says that a derived equivalence between self-injective algebras induces an equivalence of their stable categories. This result actually says that a stable equivalence induced from a derived equivalence between self-injective algebras has a special form. More precisely, suppose that A and B are two self-injective algebras which are derived equivalent, then there are two bimodules ${}_AM_B$ and ${}_BN_A$ which are projective as left modules and as right modules such that we have bimodule isomorphisms:

$${}_AM \otimes_B N_A \cong {}_AA_A \oplus {}_AP_A, \quad {}_BN \otimes_A M_B \cong {}_BB_B \oplus {}_BQ_B$$

where ${}_AP_A$ and ${}_BQ_B$ are projective bimodules. Tensoring with M or N provides stable equivalences, which Broué [4] called "stable equivalences of Morita type". Many stable equivalences constructed in modular representation theory happen to be of Morita type (see, for example, [4,19,36]).

The Auslander-Reiten conjecture asserts that if the stable categories of two Artin algebras are equivalent, then the algebras have the same number of isomorphism classes of non-projective simple modules (cf. [1, Page 409]). This conjecture has been studied by many authors (see, for instance, [26,27,30,37]) and it has been established for some special classes of algebras. Based on a complete classification of representation-finite self-injective algebras, the conjecture was solved for this class of algebras in [34]. Subsequently, Martinez-Villa [26] proved the conjecture for all algebras of finite representation type. Pogorzarły [30] gave a proof of this conjecture for self-injective special biserial algebras, and Tang [37] proved it for some class of radical cube zero self-injective algebras. All of these proofs rely heavily on the knowledge of the Auslander-Reiten quiver of the algebras in question. Martinez-Villa [27] reduced the general problem to the case where both algebras are self-injective.

If two Artin algebras are derived equivalent, then they have isomorphic Hochschild (co)homology groups HH_* and HH^* for $* \geq 0$. If two Artin algebras are stably equivalent of Morita type, then they have the same global dimension and isomorphic Hochschild (co)homology groups HH_* and HH^* for $* \geq 1$ ([22,31,39]). In degree zero, it is only known that the stable centers are isomorphic. The stable center is a quotient of the 0-degree Hochschild cohomology, that is, of the center. It is not known whether the 0-degree Hochschild (co)homology groups HH_0 and HH^0 are preserved by a stable equivalence of Morita type. In this article, we will show that roughly speaking, the invariance of the dimension of the 0-degree Hochschild (co)homology group under stable equivalences of Morita type is equivalent to the Auslander-Reiten conjecture. More precisely, we will prove the following

Theorem 1.1 *Let k be an algebraically closed field. Let A and B be two finite dimensional k-algebras which are stably equivalent of Morita type. Then the following statements are equivalent.*

(1) *A and B have the same number of isomorphism classes of simple modules.*

(2) *The* 0*-degree Hochschild homology groups* $HH_0(A)$ *and* $HH_0(B)$ *have the same dimension* (*over the ground field* k).

If A and B have no semisimple direct summands, these two conditions are further equivalent to the following

(3) *A and B have the same number of non-isomorphic non-projective simple modules.*

We remark that there are also many stable equivalences of Morita type between non-self-injective algebras [22–24]. We will give an example showing that the assumption of the stable equivalence to be of Morita type cannot be dropped; there are stably equivalent algebras that have non-isomorphic 0-degree Hochschild homology groups.

Denote by $Z(A)$ the center of an algebra A. It can be considered as the set of homomorphisms as $A^e := A \otimes_k A^{\mathrm{op}}$-modules from A to A, that is, $Z(A) = \mathrm{Hom}_{A^e}(A, A)$. The projective center $Z^{\mathrm{pr}}(A) \subset Z(A)$ is, by definition, the subset of homomorphisms which factor through projective A^e-modules. The stable center $Z^{\mathrm{st}}(A)$ is defined to be the quotient $Z(A)/Z^{\mathrm{pr}}(A)$.

For symmetric algebras, we get further equivalent versions of the Auslander-Reiten conjecture, refining the characterisation given in Theorem 1.1.

Corollary 1.2 *Suppose that in the setup of Theorem* 1.1*, one of the algebras A or B is symmetric. Then the following statements are equivalent.*

(1) *A and B have the same number of isomorphism classes of simple modules.*
(2) *The* 0*-degree Hochschild homology groups* $HH_0(A)$ *and* $HH_0(B)$ *have the same dimension.*
(3) *The centers* $Z(A)$ *and* $Z(B)$ *have the same dimension.*
(4) *The projective centers* $Z^{\mathrm{pr}}(A)$ *and* $Z^{\mathrm{pr}}(B)$ *have the same dimension.*

If A and B have no semisimple summands, these conditions are further equivalent to the following

(5) *A and B have the same number of non-isomorphic non-projective simple modules.*

This result indicates that the main obstruction to the Auslander-Reiten conjecture lies in the projective center; this concept will be studied in detail in the first part of this article. The statements of the corollary and further details to be given later, in particular on Cartan matrices, may lead to applications in modular representation theory. In a subsequent paper of the first two authors joint with Steffen König [15], we continue the study in this paper and found some more invariants detecting the validity of the Auslander-Reiten conjecture. Some applications of the results of this paper are already achieved: In a recent paper [40] the last two authors proved the Auslander-Reiten conjecture for stable equivalences of Morita type between the algebras of dihedral, semidihedral or quaternion type in the sense of Erdmann [7]. The main tool there, in particular for semidihedral and quaternion type, are Theorem 1.1 and its Corollary. Moreover, again using at essential points Theorem 1.1 and its corollary, in [41] the last two authors show that the Auslander-Reiten conjecture holds for a stable equivalence of Morita type between two indecomposable tame symmetric algebras with only periodic modules, as well as for indecomposable tame symmetric algebras of polynomial growth.

The key point of the proof of the main theorem is to define the concept of the 0-degree stable Hochschild homology group $HH_0^{\mathrm{st}}(A)$ which is analogous to its counterpart in Hochschild cohomology, the stable center $Z^{\mathrm{st}}(A) = Z(A)/Z^{\mathrm{pr}}(A)$. This stable Hochschild homology group $HH_0^{\mathrm{st}}(A)$ is a subspace of the usual 0-degree Hochschild homology group $HH_0(A)$ and its definition uses the Hattori-Stallings trace map. We then prove that $HH_0^{\mathrm{st}}(A)$ is invariant under stable equivalences of Morita type. Moreover, we deduce the equality

$$\dim HH_0^{\mathrm{st}}(A) + \mathrm{rank}_p C_A = \dim HH_0(A),$$

where C_A is the Cartan matrix and where $\mathrm{rank}_p C_A$ is its p-rank, that is, the rank of C_A over k. By analyzing the stable Grothendieck group, we are able to prove that the Auslander-Reiten conjecture for stable equivalences of Morita type is equivalent to the invariance of the p-rank of the Cartan matrices. The theorem thus follows.

We will also study the notions of the projective center and the p-rank of the Cartan matrix for Frobenius algebras. More precisely, let A be a Frobenius k-algebra with a non-degenerate associative bilinear form $(\,,\,) : A \times A \to k$. Let $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ be a pair of dual bases of A, that is, $(a_i, b_j) = \delta_{ij}$. We define two linear maps $\tau : A \to A$ by $\tau(x) = \sum_{i=1}^n b_i x a_i$ and $\theta : A \to A$ by $\theta(x) = \sum a_i x b_i$, respectively.

Proposition 1.3 *Suppose that k is an algebraically closed field and that A is a Frobenius k-algebra. Then we have the following.*

(1) *The projective center $Z^{\mathrm{pr}}(A)$ is equal to the image $\mathrm{Im}(\tau)$ of the above map τ.*
(2) *The p-rank of the Cartan matrix C_A is equal to the dimension of the image $\mathrm{Im}(\theta)$ of the above map θ.*
(3) *${}^{\perp}\mathrm{Im}(\theta)/K(A) = HH_0^{\mathrm{st}}(A)$, where ${}^{\perp}\mathrm{Im}(\theta)$ is the left orthogonal of $\mathrm{Im}(\theta)$ in A and $K(A)$ is the commutator subspace of A.*

This article is organized as follows. In the second section, we study the notion of the projective center, or equivalently, the Higman ideal. We define the stable Hochschild homology group in the third section. The fourth section contains an analysis of the stable Grothendieck groups. The main theorem and its corollary are proved in the fifth section. The last section contains an alternative proof of the main result in case that the ground field has positive characteristic.

2 Higman ideal and projective center

Throughout this paper, we denote by k a commutative ring. In certain cases we shall need to restrict the nature of k further. All algebras considered are Noetherian k-algebras with identity which are k-free of finite rank as k-module. When k is a field, they are just finite dimensional k-algebras over k. By a module over a k-algebra, we always mean a finitely generated unitary left module, unless stated otherwise. Given a k-algebra A, we give the list of some usual notations.

- A-mod the category of left A-modules
- A-$\underline{\mathrm{mod}}$ the stable category of A-mod
- $l(A)$ the number of isomorphism classes of simple A-modules
- $Z(A)$ the center of A
- $J(A)$ the Jacobson radical of A
- $\mathrm{Soc}(A)$ the socle of A
- $R(A) := \mathrm{Soc}(A) \cap Z(A)$ the Reynolds ideal of A
- $A^e := A \otimes_k A^{op}$ the enveloping algebra of A

Recall that the stable category A-$\underline{\mathrm{mod}}$ is defined as follows: the objects of A-$\underline{\mathrm{mod}}$ are the same as those of A-mod, and the morphisms between two objects X and Y are given by the quotient k-module $\underline{\mathrm{Hom}}_A(X, Y) = \mathrm{Hom}_A(X, Y)/\mathcal{P}(X, Y)$, where $\mathcal{P}(X, Y)$ is the k-submodule of $\mathrm{Hom}_A(X, Y)$ consisting of those homomorphisms from X to Y which factor through a projective A-module. Two k-algebras A and B are said to be stably equivalent if there is an equivalence $F : A\text{-}\underline{\mathrm{mod}} \to B\text{-}\underline{\mathrm{mod}}$ between the stable categories.

Note that we can identify any A-A-bimodule with an A^e-module. As we have a canonical k-algebra isomorphism $\mathrm{End}_{A^e}(A, A) \simeq Z(A)$ $(f \mapsto f(1))$, for each A^e-module V, the group $\mathrm{Ext}^1_{A^e}(A, V)$ has a natural right $Z(A)$-module structure defined as follows. We interpret elements of $\mathrm{Ext}^1_{A^e}(A, V)$ as equivalence classes of extensions of V by A. Let $\xi \in \mathrm{Ext}^1_{A^e}(A, V)$ correspond to the class of a short exact sequence of A^e-modules

$$\xi : 0 \longrightarrow V \longrightarrow X \xrightarrow{\varphi} A \longrightarrow 0.$$

Given any $a \in \mathrm{End}_{A^e}(A, A)$, the class ξa corresponds to the upper short exact sequence of the pullback diagram:

$$\begin{tikzcd}
\xi a: 0 \arrow[r] & V \arrow[r] \arrow[d, "1"'] & X_1 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "a"'] & 0 \\
\xi: 0 \arrow[r] & V \arrow[r] & X \arrow[r, "\varphi"] & A \arrow[r] & 0.
\end{tikzcd}$$

In analogy to the Higman ideal in the study of orders (see [10] and [35]), we give the following

Definition 2.1 Let k be a commutative ring and A be a k-algebra. Define the *Higman ideal* to be

$$H(A) := \{a \in Z(A) |\ \mathrm{Ext}^1_{A^e}(A, V) \cdot a = 0, \forall V \in A^e - \mathrm{mod}\}$$

Remark 2.2 Clearly $H(A)$ is an ideal of $Z(A)$. Denote by D_{univ} the following universal exact sequence

$$0 \longrightarrow \Omega_{A^e}(A) \longrightarrow A^e \longrightarrow A \longrightarrow 0,$$

where $A^e \longrightarrow A$ is given by the multiplication map. Then it is easy to show that the definition of the Higman ideal can be simplified as follows

$$H(A) := \left\{a \in Z(A) |\ D_{\mathrm{univ}} \cdot a = 0 \in \mathrm{Ext}^1_{A^e}(A, \Omega_{A^e}(A))\right\}.$$

Set $Z^{\mathrm{pr}}(A)$ to be the subset of Z(A) consisting of A^e-homomorphisms from A to A which factor through a projective A^e-module. Clearly $Z^{\mathrm{pr}}(A)$ is an ideal of $Z(A)$ and we call it the projective center of A. It is now easily observed that the Higman ideal is nothing else but the projective center $Z^{\mathrm{pr}}(A)$.

Proposition 2.3 *Let k be a commutative ring and A a k-algebra. Then we have $H(A) = Z^{\mathrm{pr}}(A)$.*

Proof In fact, define the k-linear homomorphism

$$\mu : \mathrm{Hom}_{A^e}(A, A^e) \otimes_{A^e} A \longrightarrow \mathrm{End}_{A^e}(A), \quad \mu(f \otimes a) : a' \mapsto f(a')a$$

where $a' \in A$, $f \in \mathrm{Hom}_{A^e}(A, A^e)$, $a \in A$. Using [6, Proposition 29.15] by taking $M = A$ and $\Lambda = A^e$, we see that the image of μ coincides with $H(A)$.

Since we have a canonical k-isomorphism

$$\mathrm{Hom}_{A^e}(A, A^e) \otimes_{A^e} \mathrm{Hom}_{A^e}(A^e, A) \simeq \mathrm{Hom}_{A^e}(A, A^e) \otimes_{A^e} A,$$

then

$$H(A) = \mathrm{Im}(\mu) = \{g \circ f \in \mathrm{End}_{A^e}(A) \mid f \in \mathrm{Hom}_{A^e}(A, A^e), g \in \mathrm{Hom}_{A^e}(A^e, A)\}.$$

Since any $f \in \mathrm{Hom}_{A^e}(A, A)$ factors through a projective A^e-module if and only if f factors through the regular A^e-module A^e, we know that $H(A)$ is just another formulation of the projective center $Z^{\mathrm{pr}}(A)$. □

For a symmetric k-algebra A over a field k with symmetrizing bilinear form $(,)$, Héthelyi et al. (see [9]) have defined the Higman ideal $H(A)$ as the image of the following map

$$\tau : A \to A, x \mapsto \sum_{i=1}^{n} b_i x a_i,$$

where $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are a pair of dual bases of A. This definition can be generalized to the case of a Frobenius k-algebra over a field k. Recall that a Frobenius k-algebra A is endowed with a non-degenerate associative bilinear form $(,)$: $A \times A \to k$. Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be a pair of dual bases of A defined by the relations $(a_i, b_j) = \delta_{ij}$, where δ_{ij} is the Kronecker symbol. Then it is easy to see that the image of the above map τ is independent of the choice of dual bases. So we can define the Higman ideal for a Frobenius k-algebra A as in [9]. Note that for non-symmetric Frobenius k-algebras, we can not reverse the order of the dual bases in the definition of $\tau : A \to A$ in general (see Example 2.9).

Next we show that the above definition for Frobenius k-algebras coincides with our Definition 2.1. We are grateful to the referee who suggested the following proof which is much simpler than the one proposed in the original version.

Proposition 2.4 *Suppose that k is a field and that A is a Frobenius k-algebra. Then the Higman ideal $H(A)$ in Definition 2.1 is equal to the image* $\mathrm{Im}(\tau)$ *of the above map τ.*

Proof A key observation is that the map $f : A \to A \otimes_k A$ sending 1 to $\sum_i b_i \otimes a_i$ is an injective homomorphism of bimodules. In fact, the injectivity is easy as the elements a_i form a basis and the bilinear form is non degenerate. Suppose that for $a \in A$, $ab_i = \sum_j \lambda_{ij} b_j$, then it is easy to see that $a_i a = \sum_j \lambda_{ij} a_j$. So

$$\sum_i ab_i \otimes a_i = \sum_{ij} \lambda_{ij} b_j \otimes a_i = \sum_{ij} \lambda_{ji} b_i \otimes a_j = \sum_i b_i \otimes a_i a.$$

As A is self-injective, then each homomorphism of bimodules from A to itself factoring through a projective bimodule actually factor through f. Notice furthermore that any bimodules homomorphism $A \otimes_k A \to A$ is of the form $g_x : b \otimes a \mapsto bxa$ for some $x \in A$. Therefore, the image of 1 under the composition of f with g_x is just $\tau(x)$. This proves that $\mathrm{Im}(\tau) = Z^{pr}(A)$. □

Remark 2.5 In the case of a symmetric algebra A over a field k, Broué has proven that $Z^{\mathrm{pr}}(A) = \mathrm{Im}(\tau)$, where $\tau : A \to A$ is defined as above ([5, 3.13 Proposition]).

For the completeness of our discussion, we should mention that Héthelyi et al. (see [9]) obtained another description of the Higman ideal for a symmetric k-algebra over a field k. To state it one needs some notations. Let A be a symmetric k-algebra over a field k. Then A is endowed with a non-degenerate associative symmetric bilinear form $(,)$. With respect to this bilinear form, one can define the orthogonal $V^{\perp}$ of a subspace $V \subseteq A$ by $V^{\perp} := \{a \in A \mid (a, b) = 0, \ \forall\, b \in V\}$. One proves easily ([17]) that $J(A)^{\perp} = \mathrm{Soc}(A)$

and $K(A)^{\perp} = Z(A)$, where $K(A)$ is the k-subspace of A spanned by all commutators $ab - ba\,(a, b \in A)$. Moreover for the map $\tau : A \to A$, we have $\mathrm{Im}(\tau) \subseteq R(A)$ and $J(A) + K(A) \subseteq \mathrm{Ker}(\tau)$ ([9, Lemma 4.1]).

Let $a_1 = e_1, a_2 = e_2, \dots, a_l = e_l$ be a set of representatives of conjugacy classes of primitive idempotents in A (this means that $Ae_1, \dots, Ae_l$ are representatives for the isomorphism classes of indecomposable projective left A-modules). Let $a_{l+1}, \dots, a_n$ denote a basis of $J(A)+K(A)$. Then we can prove that $a_1, \dots, a_n$ is a basis of A. Let $b_1, \dots, b_n$ be the dual basis. Then $r_1 = b_1, \dots, r_l = b_l$ is a basis of $(J(A) + K(A))^{\perp} = \mathrm{Soc}(A) \cap Z(A) = R(A)$. As $J(A) + K(A) \subset \mathrm{Ker}(\tau)$, $\mathrm{Im}(\tau)$ is k-spanned by $\tau(e_i)$ with $1 \leq i \leq l$. It follows that we only need to know $\tau(e_i)$ $(1 \leq i \leq l)$ for computing the Higman ideal.

Lemma 2.6 [9, Lemma 4.3] *With the notations above, for* $1 \leq i \leq l$,

$$\tau(e_i) = \sum_{j=1}^{l} (\dim e_i A e_j) \cdot r_j.$$

Before stating some interesting application of this lemma, we first recall the definition of the Cartan matrix. Let A be a finite dimensional k-algebra over a field k. Suppose that $P_1 = Ae_1, \dots, P_l = Ae_l$ are representatives for the isomorphism classes of indecomposable projective left A-modules and that $S_1, \dots, S_l$ are the corresponding simple modules. By definition, the Cartan matrix $C_A = (c_{ij})_{i,j=1}^{l}$ of A is a $l \times l$ integer matrix, where c_{ij} is given by the number of composition factors of P_j which are isomorphic to S_i. It is well known that $c_{ij} = \dim e_i A e_j / \dim \mathrm{End}_A(S_i)$ and therefore $c_{ij} = \dim e_i A e_j$ over an algebraically closed field k. Let $p \geq 0$ be the characteristic of k. If C_A is the Cartan matrix of A, then we denote the p-rank of C_A by $\mathrm{rank}_p(C_A)$ which, by definition, is the rank of the Cartan matrix over k. Of course, when $p = 0$ this is just the usual rank of C_A.

Corollary 2.7 *If A is a symmetric algebra over an algebraically closed field k of characteristic $p \geq 0$, then* $\mathrm{rank}_p(C_A) = \dim H(A)$.

Remark 2.8 If A is a symmetric algebra over an algebraically closed field k of characteristic 0, then clearly $\mathrm{rank}_p(C_A) \neq 0$. This implies that the Higman ideal $H(A)$ and therefore the projective center $Z^{\mathrm{pr}}(A)$ are always nonzero in this case.

Example 2.9 Let k be an algebraically closed field. Consider the 4-dimensional local Frobenius k-algebra

$$A = k\langle x, y\rangle/(x^2, y^2, xy - ryx),\ 0 \neq r \in k.$$

One can define a non-degenerate associative bilinear form over A by posing

$$(x, y) = (1, xy) = (xy, 1) = 1, (y, x) = 1/r,$$
$$(1, 1) = (1, x) = (x, 1) = (1, y) = (y, 1) = 0$$

(for a construction of such bilinear forms for weakly symmetric algebras, see Proposition 3.1 of [11]). Note that A is symmetric if and only if $r = 1$. It follows that $\{a_1 = 1, a_2 = x, a_3 = y, a_4 = xy\}$ and $\{b_1 = xy, b_2 = y, b_3 = rx, b_4 = 1\}$ are a pair of dual bases of A. Then $\mathrm{Im}(\tau) = \langle (2 + r + \frac{1}{r})xy\rangle$, the subspace generated by $(2 + r + \frac{1}{r})xy$. However, if we define $\theta : A \to A$ by $x \mapsto \sum_{i=1}^{n} a_i x b_i$, then $\mathrm{Im}(\theta) = \langle 4xy\rangle$. So, if $\mathrm{char}(k) \neq 2$, the two images are different for $r = -1$; if $\mathrm{char}(k) = 2$, the two images are different for all r with $r \neq 1$. On the other hand, the Cartan matrix of A is a 1×1 matrix (4). Notice that if char$(k) \neq 2$ and $r = -1$ or if char$(k) = 2$ and $r \neq 1$, then $\mathrm{rank}_p(C_A) \neq \dim H(A)$. This means that Corollary 2.7 is in general not true for non-symmetric algebras. Clearly in this example $\mathrm{rank}_p(C_A) = \dim \mathrm{Im}(\theta)$. This is not a coincidence. In fact, this is true for arbitrary Frobenius algebras (Proposition 3.16).

Finally, we state some useful properties of the Higman ideal. We thank Shengyong Pan for pointing out an inaccuracy in the original proof of the following proposition.

Proposition 2.10 *Suppose that B and C are two k-algebras over a commutative ring k.*

(1) *Put $A = B \times C$. Then $H(A) = H(B) \times H(C)$ and this decomposition is compatible with the decomposition of centers $Z(B \times C) = Z(B) \times Z(C)$.*
(2) *If B and C are two derived equivalent algebras which are projective over k as k-modules, then there is an algebra isomorphism of centers of B and C mapping $H(B)$ to $H(C)$.*

Proof (1) By Proposition 2.3, the Higman ideal is equal to the projective center. But the latter clearly satisfies the decomposition property.
(2) Since the Higman ideal is equal to the projective center, it suffices to show that the projective center is a derived invariant. The latter has been proven by Broué [4, 4.4 Corollary] for self-injective algebras. In fact, the idea in Broué's proof can be applied to arbitrary k-algebras which are projective over k. For the convenience of the reader, we shall give the detailed proof here.

By a result of Rickard [33] (see also [16]), there is a complex $X^\bullet \in D^b(B \otimes_k C^{\mathrm{op}})$ so that

$$X^\bullet \otimes^{\mathbb{L}}_C - : D^b(C) \longrightarrow D^b(B)$$

is an equivalence as triangulated categories. The (bounded) complex $X^\bullet$ is called a two-sided tilting complex if it satisfies the following equalities in $D^b(B^e)$ and in $D^b(C^e)$ respectively,

$$X^\bullet \otimes^{\mathbb{L}}_C \mathrm{Hom}_B(X^\bullet, B) \simeq B, \ \ \mathrm{Hom}_B(X^\bullet, B) \otimes^{\mathbb{L}}_B X^\bullet \simeq C.$$

Now since B and C are projective over k, we may assume that $X^\bullet$ is a complex of bimodules each of which is projective as left-modules and projective as right-modules. Therefore the left derived tensor product can be replaced by the ordinary tensor product, which then is associative. It follows that tensor product with $X^\bullet$ from the left and with $\mathrm{Hom}_B(X^\bullet, B)$ from the right is associative and

$$X^\bullet \otimes_C - \otimes_C \mathrm{Hom}_B(X^\bullet, B) : D^b(C^e) \longrightarrow D^b(B^e)$$

is an equivalence between the derived categories of enveloping algebras over C and B. Under this equivalence, the C^e-module C corresponds to B^e-module B, and the C^e-homomorphisms from C to C correspond to B^e-homomorphisms from B to B. Moreover, any C^e-homomorphism f from C to C factoring through a projective C^e-module corresponds to a B^e-homomorphism g from B to B factoring through a complex $P^\bullet$ of projective B^e-modules in $D^b(B^e)$. But the homomorphism from $P^\bullet$ to B can be seen as a homomorphism in $K^b(B^e)$ and it factors through the canonical multiplication map $B^e \longrightarrow B$. It follows that g is a homomorphism which factors through the regular B^e-module B^e and therefore g lies in $Z^{\mathrm{pr}}(B)$. Thus there is an algebra isomorphism of centers of B and C mapping $Z^{\mathrm{pr}}(B)$ to $Z^{\mathrm{pr}}(C)$.

3 The stable Hochschild homology

Let A and B be two algebras over a commutative ring k. Let M be an A-B-bimodule such that M_B is finitely generated and projective. Then M_B is isomorphic to a direct summand of some free right B-module B^s and there exist elements $m_i \in M$ and $\varphi_i \in \mathrm{Hom}_B(M, B)$ with $1 \le i \le s$ such that for each $m \in M$, $m = \sum_{i=1}^s m_i\varphi_i(m)$. We define the transfer map as follows.

$$t_M : A/K(A) \to (M \otimes_B \mathrm{Hom}_B(M, B))/[A, M \otimes_B \mathrm{Hom}_B(M, B)] \to B/K(B)$$
$$a \mapsto \textstyle\sum_{i=1}^s am_i \otimes \varphi_i \mapsto \sum_{i=1}^s \varphi_i(am_i),$$

where $[A, M \otimes_B \mathrm{Hom}_B(M, B)] = \{ax - xa \mid a \in A, x \in M \otimes_B \mathrm{Hom}_B(M, B)\}$. Notice that this map is just the composition of a map induced by the ring homomorphism $A \to \mathrm{End}_B(M) \simeq M \otimes_B \mathrm{Hom}_B(M, B)$ and a map induced by the Hattori-Stallings trace map $\mathrm{End}_B(M) \to B/K(B)$.

We summarize some well known properties of the transfer map in the following

Lemma 3.1 ([3, Section 3]) *Let A, B and C be k-algebras over a commutative ring k.*

(1) *If M is an A-B-bimodule and N is a B-C-bimodule such that M_B and N_C are finitely generated and projective, then we have $t_N \circ t_M = t_{M\otimes_B N}$.*
(2) *Let*

$$0 \to L \to M \to N \to 0$$

be a short exact sequence of A-B-bimodules which are finitely generated and projective as right B-modules. Then $t_M = t_L + t_N$.
(3) *Consider A as an A-A-bimodule by left and right multiplications, then t_A is the identity map.*

Example 3.2 (1) Let k be a field and A be a finite dimensional k-algebra. Let $e \in A$ be an idempotent. Considering an indecomposable projective A-module Ae as an A-k-bimodule, we have the transfer map $t_{Ae} : A/K(A) \to k$. Choose a basis $\{x_1, \dots, x_s\}$ of Ae. Then its dual basis $\{x_1^*, \dots, x_s^*\} \subset (Ae)^* = \mathrm{Hom}_k(Ae, k)$. By construction, $t_{Ae}(a + K(A)) = \sum_{i=1}^s x_i^*(ax_i)$. Observe that $\sum_{i=1}^s x_i^*(ax_i)$ is just the trace of the linear map from Ae to Ae induced by the left multiplication by a.

(2) Let A be a Frobenius k-algebra with a non-degenerate associative bilinear form $(\ ,\ ) : A \times A \to k$. Let $\{a_i\}$ and $\{b_i\}$ be a pair of dual bases of A, that is, $(a_i, b_j) = \delta_{ij}$. Note that in this case $\{(\ , b_i) =: a_i^*\} \subset A^* = \mathrm{Hom}_k(A, k)$ gives the usual dual basis of $\{a_i\}$. We define a linear map $\theta : A \to A$ by $\theta(a) = \sum a_i a b_i$ for any $a \in A$ (cf. next section). Let $Ae \subseteq A$ be a projective A-module. We can choose $\{a_i\}$ as a union of a basis of Ae and that of $A(1-e)$. By (1), $t_{Ae}(a + K(A)) = \sum_{a_i \in Ae} a_i^*(aa_i)$. On the other hand, $(a, \theta(e)) = (a, \sum a_i e b_i) = \sum (aa_i e, b_i) = \sum_{a_i \in Ae}(aa_i, b_i) = \sum_{a_i \in Ae} a_i^*(aa_i)$. It follows that $t_{Ae}(a + K(A)) = (a, \theta(e))$.

Remark 3.3 In the definition of trace map given above, we use an A-B-bimodule M which is finitely generated and projective as a right B-module. We can also define another transfer map $t_M : A/K(A) \to B/K(B)$ for a bimodule ${}_BM_A$ which is finitely generated and projective as a left B-module. The definition is similar and we omit it.

Remark 3.4 The construction of transfer maps was generalized to higher degree Hochschild homology by Bouc [3] and Keller [13]. We shall use and refer to this generalization in a subsequent paper [15]. For transfer maps in higher degree Hochschild homology groups, except the properties in Lemma 3.1, there is an additional one:

(4) Suppose that k is an algebraically closed field and that A and B are finite dimensional k-algebras. Then for a finitely generated projective A-B-bimodule P, the transfer map $t_P : HH_n(A) \to HH_n(B)$ is zero for each $n > 0$.

Let k be an algebraically closed field and let A and B be two finite dimensional k-algebras. Suppose that two bimodules M and N define a stable equivalence of Morita type between A and B by $M \otimes_B N \simeq A \oplus P, \quad N \otimes_A M \simeq B \oplus Q$. Since $t_A = 1_{HH_n(A)}$ and $t_B = 1_{HH_n(B)}$, the transfer maps $t_M : HH_n(A) \to HH_n(B)$ and $t_N : HH_n(B) \to HH_n(A)$ are mutually inverse group isomorphisms for all $n > 0$ by the properties (1)-(4). This gives a different approach for the result in [22, Theorem 4.4]. We also notice that Example 3 of [22, Section 5] is in fact not a counterexample for the non-invariance of the 0-degree Hochschild homology group since the commutator subspace $K(A) = [A, A]$ of A is not an ideal in general.

From now on, k denotes a field and all algebras are supposed to be finite dimensional over k.

In the following, we define a dual notion of the stable center and prove that it is invariant under stable equivalences of Morita type. We remind the reader that the invariance of the stable centers under stable equivalences of Morita type has been shown by Broué ([4]).

Given a k-algebra A, recall that the center $Z(A)$ of A is equal to the 0-degree Hochschild cohomology algebra $HH^0(A)$ and the stable center $Z^{\mathrm{st}}(A)$ is defined to be the quotient algebra $Z(A)/Z^{\mathrm{pr}}(A)$ where $Z^{\mathrm{pr}}(A)$ is the projective center. Motivated by this fact, we introduce the following

Definition 3.5 Let A be a finite dimensional k-algebra over a field k with the decomposition ${}_AA = \bigoplus_{i=1}^r Ae_i$, where Ae_i $(1 \le i \le r)$ are indecomposable projective A-modules. We define the (left) stable Hochschild homology group $HH_0^{\mathrm{st}}(A)$ of degree zero to be a subgroup of the 0-degree Hochschild homology group $HH_0(A) = A/K(A)$, namely

$$HH_0^{\mathrm{st}}(A) = \bigcap_{i=1}^{r} \mathrm{Ker}\{t_{Ae_i} : A/K(A) \to k\},$$

where t_{Ae_i} is the transfer map determined by the projective A-k-bimodule Ae_i.

Remark 3.6 By Example 3.2 (1), we have

$$HH_0^{\mathrm{st}}(A) = \{a \in A \mid \text{ the trace of the map } Ae_i \to Ae_i (b \mapsto ab) \\ \text{vanishes for any } 1 \le i \le r\}/K(A).$$

Remark 3.7 In the above definition, we used the transfer maps defined by the left indecomposable modules Ae_i considered as right k-modules. If we use the transfer maps defined by the right indecomposable projective modules e_iA as in Remark 3.3, then we have a notion of right stable Hochschild homology of degree zero. In the following, we shall mainly study the left Hochschild homology of degree zero. The corresponding properties of the right version are similar and the necessary modifications are left to the reader.

Theorem 3.8 *Let A be a finite dimensional k-algebra over an algebraically closed field k of characteristic $p \ge 0$. Then*

$$\dim HH_0^{\mathrm{st}}(A) + \mathrm{rank}_p(C_A) = \dim HH_0(A).$$

Proof By Lemma 3.1 (2), if two indecomposable projective modules Ae_i and Ae_j are isomorphic, then $t_{Ae_i} = t_{Ae_j} : A/K(A) \to k$. Since $HH_0(A)$ is invariant under Morita equivalences, so is $HH_0^{\mathrm{st}}(A)$. Now all terms in the assertion are Morita invariant. So we can assume that A is basic. Now $K(A) \subseteq J(A)$ and by the Wedderburn-Malcev theorem, $A = J(A) \oplus \bigoplus_i ke_i$, where $\{e_1, \ldots, e_l\}$ is a complete list of orthogonal primitive idempotents. So we can take a basis of $A/K(A)$ consisting all e_i and the classes of some elements of $J(A)$ in $J(A)/K(A)$.

Since each element a in $J(A)$ is nilpotent, for each j, the trace of the map $Ae_j \to Ae_j,\ b \mapsto ab$ is zero; now for $1 \le i, j \le l$, the trace of the map $Ae_j \to Ae_j,\ b \mapsto e_ib$ is the dimension of the space e_iAe_j, which is the Cartan invariant c_{ij}. We have proved that the matrix of the map $t_{Ae_i} : A/K(A) \to k$ under the chosen basis of $A/K(A)$ is the vector $(c_{1i}, c_{2i}, \ldots, c_{li}, 0, \ldots, 0)$ where c_{ij} is the (i, j)-entry of the Cartan matrix C_A and therefore the matrix of $(t_{Ae_1}, \ldots, t_{Ae_l}) : A/K(A) \to k^l$ is equal to $(C_A^T, 0)$. The result thus follows.

Corollary 3.9 *Let A and B be two derived equivalent finite dimensional algebras over an algebraically closed field. Then*

$$\dim HH_0^{\mathrm{st}}(A) = \dim HH_0^{\mathrm{st}}(B).$$

Proof This follows easily from Theorem 3.8 and the fact that both the p-rank of the Cartan matrix and the 0-degree Hochschild homology group are invariant under a derived equivalence. Indeed, a derived equivalence induces a mapping on the Grothendieck groups of the algebras. It is well-known that this mapping commutes with the Cartan mapping. Hence, the same holds tensoring the Grothendieck groups with the base field over the integers. This shows that the p-rank of the Cartan matrix is a derived invariant. □

Remark 3.10 (1) The proof of Theorem 3.8 implies we have a short exact sequence

$$0 \to (J(A) + K(A))/K(A) \to HH_0^{\mathrm{st}}(A) \to \mathrm{Ker}(C_A^T) \to 0,$$

where $\mathrm{Ker}(C_A^T)$ is the kernel of the linear map from $k^{l(A)}$ to itself defined by the transpose of the Cartan matrix. If we consider the right version of the stable Hochschild homology, then there is a short exact sequence

$$0 \to (J(A) + K(A))/K(A) \to HH_0^{\mathrm{st}}(A) \to \mathrm{Ker}(C_A) \to 0.$$

(2) It is well known that the 0-degree Hochschild homology group $HH_0(A)$ has a natural $Z(A)$-module structure. At least for k an algebraically closed field, we can show that the stable Hochschild homology group $HH_0^{\mathrm{st}}(A)$ is a $Z(A)$-submodule of $HH_0(A)$. In fact, we can assume that A is indecomposable and thus $Z(A)$ is local. Let $x \in HH_0^{\mathrm{st}}(A)$. Then $t_{Ae_i}(x) = 0$ for each i. Let $z \in Z(A)$, write $z = \lambda + u$ where $\lambda \in k$ and u is nilpotent. Now $t_{Ae_i}(zx) = t_{Ae_i}(\lambda x) + t_{Ae_i}(ux)$. We have $t_{Ae_i}(\lambda x) = \lambda t_{Ae_i}(x) = 0$ and since u is nilpotent, the left multiplication by ux on Ae_i is a nilpotent linear map, $t_{Ae_i}(ux) = 0$. Therefore $t_{Ae_i}(zx) = 0$ for each i and $zx \in HH_0^{\mathrm{st}}(A)$.

Theorem 3.11 *Suppose that there is a stable equivalence of Morita type between two finite dimensional k-algebras A and B over an algebraically closed field k. Then their stable Hochschild homology groups $HH_0^{\mathrm{st}}(A)$ and $HH_0^{\mathrm{st}}(B)$ are isomorphic.*

Proof Suppose that the stable equivalence of Morita type between A and B are defined by

$$M \otimes_B N \simeq A \oplus P, \quad N \otimes_A M \simeq B \oplus Q.$$

Then we have the transfer maps $t_M : HH_0(A) \to HH_0(B)$ and $t_N : HH_0(B) \to HH_0(A)$. Moreover,

$$t_N \circ t_M = t_{M\otimes_B N} = t_A + t_P = 1_{HH_0(A)} + t_P : HH_0(A) \to HH_0(A).$$

First we show that the restriction of t_P to the stable Hochschild homology group $HH_0^{\mathrm{st}}(A)$ is zero, that is, $t_P \mid_{HH_0^{\mathrm{st}}(A)} = 0$. Since P is a projective A-A-bimodule and therefore is isomorphic to a direct sum of bimodules of the form $Ae_i \otimes_k e_j A$, where e_i and e_j are primitive idempotents in A. By definition of $HH_0^{\mathrm{st}}(A)$, we know that

$$t_{Ae_i\otimes_k e_j A} \mid_{HH_0^{\mathrm{st}}(A)} = t_{e_j A} \circ t_{Ae_i} \mid_{HH_0^{\mathrm{st}}(A)} = 0$$

by Lemma 3.1. It follows that $t_P \mid_{HH_0^{\mathrm{st}}(A)} = 0$.

Next we show that $t_M(HH_0^{\mathrm{st}}(A)) \subseteq HH_0^{\mathrm{st}}(B)$. For any indecomposable projective B-module Bf, we have

$$t_{M\otimes_B Bf} = t_{Bf} \circ t_M : HH_0(A) \to HH_0(B) \to HH_0(k) = k.$$

Since $M \otimes_B Bf$ is a projective A-module, we have that $t_{M\otimes_B Bf} \mid_{HH_0^{\mathrm{st}}(A)} = 0$. This implies that $t_M(HH_0^{\mathrm{st}}(A)) \subseteq HH_0^{\mathrm{st}}(B)$. Similarly, we can show that

$$t_Q \mid_{HH_0^{\mathrm{st}}(B)} = 0 \quad \text{and} \quad t_N(HH_0^{\mathrm{st}}(B)) \subseteq HH_0^{\mathrm{st}}(A).$$

Combing our discussion above, we have proved that the transfer maps t_M and t_N induce inverse group isomorphisms between $HH_0^{\mathrm{st}}(A)$ and $HH_0^{\mathrm{st}}(B)$. □

Remark 3.12 The algebraically closed field condition is necessary in Theorem 3.11. For example, let $k = \mathbb{R}$. Consider two k-algebras $A = \mathbb{R}$ and $B = \mathbb{C}$. Since they are separable algebras, they are stably equivalent of Morita type, but we see easily that $\dim HH_0^{\mathrm{st}}(A) = 0$ and $\dim HH_0^{\mathrm{st}}(B) = 1$.

We will establish a realization of the stable Hochschild homology group for Frobenius algebras. Let A be a Frobenius k-algebra with a non-degenerate associative bilinear form $(\ ,\ ) : A \times A \to k$. Since this bilinear form is not necessarily symmetric, we have the notions of left orthogonal and right orthogonal. Let $V \subseteq A$ be a subspace. Then we define the subspaces

$$^{\perp}V := \{x \in A : (x, a) = 0, \forall\, a \in V\}, \quad V^{\perp} := \{x \in A : (a, x) = 0, \forall\, a \in V\}.$$

The following lemma is easy and its proof is left to the reader.

Lemma 3.13 (1) *If I is a left ideal of A, then $II^{\perp} = 0$.* (2) $J(A)^{\perp} = Soc(A)$.

Let $\{a_i\}$ and $\{b_i\}$ be a pair of dual bases of A, that is, $(a_i, b_j) = \delta_{ij}$. We define a linear map $\theta : A \to A$ by $\theta(x) = \sum a_i x b_i$ for any $x \in A$. It is readily seen that the definition does not depend on the choice of the dual bases. However, this map depends on the choice of the bilinear form, as can be seen in the following example given by Ohtake and Fukushima. We present it in a more general form than the original one.

Example 3.14 ([29]) Let k be an algebraically closed field of characteristic two and $A = M_2(k)$ the algebra of 2×2 matrices. For $1 \le i, j \le 2$, denote by E_{ij} the matrix whose entry at the position (i, j) is 1 and is zero elsewhere. Let P be an invertible matrix. Define two bilinear form $(\ ,\ )$ and $(\ ,\ )'$ on A by posing $(E_{ij}, E_{st}) = \delta_{it}\delta_{js}$ and $(M, N)' = (M, NP^{-1})$ for $M, N \in M_2(k)$. It is readily seen that the image of θ corresponding to the first bilinear form is the one-dimensional vector space generated by the identity matrix and the image of θ' corresponding to the second bilinear form is the one-dimensional space generated by P.

We can nevertheless prove the following

Lemma 3.15 *Let A be a Frobenius algebra with two non-degenerate associative bilinear forms $(\ ,\ )$ and $(\ ,\ )'$. Denote by θ, θ' the maps defined above corresponding to the two bilinear forms. Then* $\dim \mathrm{Im}(\theta) = \dim \mathrm{Im}(\theta')$.

Proof Indeed, there is an invertible element $u \in A$ such that $(a, b)' = (a, bu^{-1})$ for any $a, b \in A$. So if $\{x_i\}, \{y_i\}$ is a pair of dual bases for $(\ ,\ )$, then $\{x_i\}, \{y_i u\}$ is a pair of dual bases for $(\ ,\ )'$. Computing $\mathrm{Im}(\theta)$ and $\mathrm{Im}(\theta')$ using these bases, one get $\mathrm{Im}(\theta') = \mathrm{Im}(\theta)u \subseteq A$. So their dimensions are equal. □

The following proposition summarizes some properties of the above map θ. Some idea of our proof comes from [9, Section 4]. Notice that ${}^{\perp}\mathrm{Im}(\theta)$ in the following proposition is independent to the choice of bilinear forms.

Proposition 3.16 *Let A be a Frobenius algebra over an algebraically closed field k of characteristic $p \geq 0$. Then we have the following.*

(1) $\mathrm{Im}(\theta) \subseteq Soc(A)$ *and* $J(A) \subseteq \mathrm{Ker}(\theta)$.
(2) $K(A) \subseteq {}^{\perp}\mathrm{Im}(\theta)$ *and* ${}^{\perp}\mathrm{Im}(\theta)/K(A) = HH_0^{\mathrm{st}}(A)$.
(3) $\dim\ \mathrm{Im}(\theta) = \mathrm{rank}_p C_A$.
(4) *If A is basic, then under suitable bases of $A/J(A)$ and* $\mathrm{Soc}(A)$*, the matrix of the map* $\overline{\theta} : A/J(A) \to \mathrm{Soc}(A)$ *induced from θ is the transpose of the Cartan matrix C_A.*

Proof We choose carefully a basis $\{a_i\}$ and its dual basis $\{b_i\}$ in A. Suppose that

$$A/J(A) \simeq M_{u_1}(k) \times \cdots M_{u_r}(k).$$

Write E_{ij}^t for the matrix in $M_{u_t}(k)$ whose entry at the position (i, j) is 1 and is zero elsewhere. Then take $a_1 = e_1, a_2 = e_2, \ldots, a_m = e_m \in A$ such that their images in $A/J(A)$ correspond to the matrices E_{ii}^t for $1 \leq i \leq u_t$ and $1 \leq t \leq r$ and take $a_{m+1}, \ldots, a_n$ such that their images in $A/J(A)$ correspond to E_{ij}^t for $1 \leq i \neq j \leq u_t$ and $1 \leq t \leq r$. Then $\{a_1 \ldots, a_n\}$ are linearly independent in A and their images in $A/J(A)$ form a basis of $A/J(A)$ as a vector space. Notice that for $m + 1 \leq u \leq n$, $a_u \in e_i A e_j$ for some $1 \leq i \neq j \leq m$. Moreover let $a_{n+1}, \ldots, a_s \in J(A)$ such that their images in $J(A)/J^2(A)$ form a basis of $J(A)/J^2(A)$ as a vector space, let $a_{s+1}, \ldots, a_t \in J^2(A)$ such that their images in $J^2(A)/J^3(A)$ form a basis of $J^2(A)/J^3(A)$ as a vector space, etc. Let $b_1, b_2, \ldots$ be the dual basis. Then $\{r_1 = b_1, \ldots, r_n = b_n\}$ is a basis of $J(A)^{\perp} = \mathrm{Soc}(A)$, $\{b_1, \ldots, b_s\}$ is a basis of $J^2(A)^{\perp}$, $\{b_1, \ldots, b_t\}$ is a basis of $J^3(A)^{\perp}$, etc.

Now we can prove the first assertion. Since for any $x \in A$,

$$J(A)x = 0 \Longleftrightarrow x \in \mathrm{Soc}(A) \Longleftrightarrow xJ(A) = 0,$$

we need to prove that $J(A) \cdot \mathrm{Im}(\theta) = 0$. Let $y \in J(A)$ and $x \in A$. For $1 \leq i \leq n$, we get

$$ya_i x b_i \in J(A) \cdot A \cdot A \cdot \mathrm{Soc}(A) = J(A) \cdot \mathrm{Soc}(A) = 0;$$

for $n + 1 \leq i \leq s$, we get

$$ya_i x b_i \in J(A) \cdot J(A) \cdot A \cdot J^2(A)^{\perp} = J^2(A) \cdot J^2(A)^{\perp} = 0;$$

for $s+1 \leq i \leq t$, we get

$$ya_ixb_i \in J(A) \cdot J^2(A) \cdot A \cdot J^3(A)^{\perp} = J^3(A) \cdot J^3(A)^{\perp} = 0;$$

etc. This proves that $\mathrm{Im}(\theta) \subseteq \mathrm{Soc}(A)$.

Now let $x \in J(A)$. Then for $1 \leq i \leq n$, we get

$$a_ixb_i \in A \cdot J(A) \cdot \mathrm{Soc}(A) = J(A) \cdot \mathrm{Soc}(A) = 0;$$

for $n+1 \leq i \leq s$, we get

$$a_ixb_i \in J(A) \cdot J(A) \cdot J^2(A)^{\perp} = J^2(A) \cdot J^2(A)^{\perp} = 0;$$

for $s+1 \leq i \leq t$,

$$a_ixb_i \in J^2(A) \cdot J(A) \cdot J^3(A)^{\perp} = J^3(A) \cdot J^3(A)^{\perp} = 0;$$

etc. This proves that $J(A) \subseteq \mathrm{Ker}(\theta)$.

Next we come to the second and the third assertions. We prove first that $a_{m+1}, \ldots, a_n \in \mathrm{Ker}(\theta)$. We will choose another basis of A and its dual basis. Let $X_{ij} \subset e_iAe_j$ be a basis of e_iAe_j for $1 \leq i, j \leq m$. Then $X = \cup X_{ij}$ is a basis of A. If $x \in X_{ij}$, denote by x^* the corresponding element in the dual basis and it is easy to see that $x^* \in e_jA$. Now for $m+1 \leq u \leq n$, $\theta(a_u) = \sum_{x \in X} xa_ux^*$. Note that $a_u \in e_sAe_t$ for some $s \neq t$. Since $xa_ux^* \in e_iAe_je_sAe_te_jA = 0$ for $x \in X_{ij}$, we have $\theta(a_u) = 0$. This implies that $a_{m+1}, \ldots, a_n \in \mathrm{Ker}(\theta)$ and that $\theta(e_1), \ldots, \theta(e_m)$ generate $\mathrm{Im}(\theta)$ as a vector space.

Since by Example 3.2(2), $t_{Ae_i}(\overline{x}) = (x, \theta(e_i))$, for $x \in A$ and $1 \leq i \leq m$ where $\overline{x}$ is the class of x in $A/K(A)$, we have $\overline{x} \in HH_0^{\mathrm{st}}(A) \iff x \in {}^{\perp}\mathrm{Im}(\theta)$. This proves that $K(A) \subseteq {}^{\perp}\mathrm{Im}(\theta)$ and ${}^{\perp}\mathrm{Im}(\theta)/K(A) = HH_0^{\mathrm{st}}(A)$. Thus the second assertion holds. For the third assertion, by Theorem 3.8,

$$\dim(A/K(A)) = \dim({}^{\perp}\mathrm{Im}(\theta)/K(A)) + \mathrm{rank}_p C_A$$

and we see that

$$\begin{aligned} \dim(\mathrm{Im}(\theta)) &= \dim(A) - \dim({}^{\perp}\mathrm{Im}(\theta)) \\ &= \dim(A/K(A)) - \dim({}^{\perp}\mathrm{Im}(\theta)/K(A)) = \mathrm{rank}_p C_A. \end{aligned}$$

Now suppose that A is basic. As above let $e_1, \ldots, e_m$ be a complete set of orthogonal primitive idempotents and extend them to a basis of A and take the dual basis $\{b_1 = r_1, \ldots, b_m = r_m, \cdots\}$. Remark that now the images of $e_1, \ldots, e_m$ in $A/J(A)$ are a basis of $A/J(A)$ and $\{r_1, \ldots, r_m\}$ is a basis of $J(A)^{\perp} = \mathrm{Soc}(A)$. By (1), we have an induced map $\overline{\theta} : A/J(A) \to \mathrm{Soc}(A)$. Note that the source and the target of this map are both of dimension $l(A)$. We will prove that its matrix under the basis $\{e_1, \ldots, e_m\}$ of $A/J(A)$ and the basis $\{r_1, \ldots, r_m\}$ of $\mathrm{Soc}(A)$ is just the transpose of the Cartan matrix C_A. Suppose that $\overline{\theta}(e_i) = \sum \lambda_{ij} r_j$. Then $\lambda_{ij} = (e_j, \overline{\theta}(e_i)) = (1, e_j\overline{\theta}(e_i))$. To compute the latter, as above, choose a basis X_{ij} of e_iAe_j and let $X = \cup X_{ij}$. Now

$$e_j\overline{\theta}(e_i) = \sum_{x \in X} e_jxe_ix^* = \sum_{x \in X_{ji}} xx^*.$$

We have thus

$$\lambda_{ij} = (1, e_j\overline{\theta}(e_i)) = (1, \sum_{x \in X_{ji}} xx^*) = \sum_{x \in X_{ji}} (x, x^*) = \dim(e_jAe_i) \cdot 1.$$

$\square$

For a symmetric k-algebra A, we can give a simple description for the stable Hochschild homology group $HH_0^{\mathrm{st}}(A)$.

Proposition 3.17 *Let A be a finite dimensional symmetric k-algebra over an algebraically closed field k. Then we have that $HH_0^{\mathrm{st}}(A) = Z^{\mathrm{pr}}(A)^{\perp}/K(A)$, where $Z^{\mathrm{pr}}(A)^{\perp}$ is the orthogonal space of the projective center $Z^{\mathrm{pr}}(A)$.*

Proof For symmetric algebras, the map θ is the same as the map τ introduced in Sect. 2. So by Proposition 3.16 (2) and Proposition 2.3, we have that

$$HH_0^{\mathrm{st}}(A) = {}^{\perp}\mathrm{Im}(\theta)/K(A) = {}^{\perp}\mathrm{Im}(\tau)/K(A) = H(A)^{\perp}/K(A) = Z^{\mathrm{pr}}(A)^{\perp}/K(A).$$

□

Remark 3.18 (1) Let k be an algebraically closed field. If A is a finite dimensional k-algebra of finite global dimension, then $HH_0^{\mathrm{st}}(A) = 0$. In fact, in this case the determinant of the Cartan matrix is invertible in $\mathbb{Z}$, so the p-rank of the Cartan matrix is full, that is, $\mathrm{rank}_p C_A = l(A) = \dim A/(J(A) + K(A))$ where the last equality is Formula (5) of [17]. So we have the equality $HH_0^{\mathrm{st}}(A) = (J(A) + K(A))/K(A)$ by Remark 3.10 (1). On the other hand, according to [18], $(J(A) + K(A))/K(A) = 0$ since A has finite global dimension.

(2) In [8], Eu and Schedler also defined a notion of stable Hochschild (co)-homology groups for any Frobenius k-algebra. In particular, for a Frobenius algebra A over a field k, the 0-degree stable Hochschild homology group of A is defined there as the kernel of the canonical homomorphism $A \otimes_{A^e} A \to A \otimes_{A^e} I$, where I is the injective envelope of the A^e-module A. We point out that even for symmetric k-algebras, the two definitions of 0-degree stable Hochschild homology group may be different. For example, let k be a field of characteristic 0 and let $A = k[x]/(x^2)$. Then it is easy to compute that the 0-degree stable Hochschild homology group in our sense is a one dimensional k-space but it is zero in Eu and Schedler's sense.

Example 3.19 Let A be an indecomposable non-simple self-injective Nakayama algebra over an algebraically closed field k of characteristic $p \geq 0$. That is, there is a cyclic oriented quiver Q with s vertices such that $A = kQ/kQ^{\geq sn+t}$ with $n \geq 0$, $0 < t \leq s$ and $sn + t \geq 2$ and where $kQ^{\geq sn+t}$ is the ideal generated by all paths of length $\geq sn + t$. We can prove that

$$\dim HH_0^{\mathrm{st}}(A) = \begin{cases} n + u - 1, & \text{if } p \nmid \frac{sn+t}{u} \\ n + u, & \text{if } p \mid \frac{sn+t}{u}, \end{cases}$$

where u is the greatest common divisor of s and t. Notice that the dimension of the degree zero stable Hochschild homology group depends on p. In fact, since $\dim A/K(A) = s + n$, by Theorem 3.8, one only needs to compute the p-rank of the Cartan matrix C_A. This has been done in [28] which gives the following formula

$$\mathrm{rank}_p C_A = \begin{cases} s - u + 1, & \text{if } p \nmid \frac{sn+t}{u} \\ s - u, & \text{if } p \mid \frac{sn+t}{u}. \end{cases}$$

4 Cartan matrices and stable Grothendieck groups

Let k be a field and A be a finite-dimensional k-algebra. The stable Grothendieck group $G_0^{\mathrm{st}}(A)$ is by definition the cokernel of the Cartan map. In other words, we have the following short exact sequence

$$K_0(A) \overset{C_A}{\to} G_0(A) \to G_0^{\mathrm{st}}(A) \to 0,$$

where C_A is the Cartan matrix of A and where $K_0(A)$ (respectively, $G_0(A)$) is a free abelian group of finite rank generated by isomorphism classes of indecomposable projective modules (respectively, isomorphism classes of simple modules). Suppose that the invariant factors of the Cartan matrix are $\{0, \ldots, 0, 1, \ldots, 1, \delta_1, \delta_2, \ldots, \delta_r\}$, where $\delta_i \geq 2$. Denote by m_i $(i = 0, 1)$ the number of i in the above sequence. Then the stable Grothendieck group is isomorphic to $\mathbb{Z}^{m_0} \oplus \mathbb{Z}/\delta_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/\delta_r\mathbb{Z}$. We remark that the stable Grothendieck group gives the precise values of m_0 and r but cannot detect m_1 which is the number of 1 in the sequence of invariant factors of C_A.

Now we consider two finite dimensional k-algebras A and B. If they are stably equivalent of Morita type, then it is known that their stable Grothendieck groups are isomorphic (see [39, Section 5]). Note that for a self-injective k-algebra, its stable category is naturally triangulated and the stable Grothendieck group is just the Grothendieck group of the stable category as a triangulated category (see [38, Proposition 1]). Thus if the stable categories of two self-injective k-algebras are triangle equivalent, then their stable Grothendieck groups are isomorphic. Our aim is to prove the following result.

Proposition 4.1 *Let A and B be two finite dimensional k-algebras over a field k. Suppose that A and B are stably equivalent of Morita type or that they are self-injective and their stable categories are triangle equivalent. Then the following statements are equivalent.*

(1) *A and B have the same number of isomorphism classes of simple modules, that is, $l(A) = l(B)$;*
(2) $\mathrm{rank}_p(C_A) = \mathrm{rank}_p(C_B)$;
(3) *The number of 1 in the set of invariant factors of the Cartan matrix of A and that of B are the same, that is, $m_1(A) = m_1(B)$.*

Proof By the discussion above, the stable Grothendieck groups of A and B are isomorphic, so are the stable Grothendieck groups with coefficients in k, that is, $G_0^{\mathrm{st}}(A) \otimes_{\mathbb{Z}} k \simeq G_0^{\mathrm{st}}(B) \otimes_{\mathbb{Z}} k$. Tensoring by k on the short exact sequence defining the stable Grothendieck group, we have

$$K_0(A) \otimes_{\mathbb{Z}} k \xrightarrow{C_A \otimes_{\mathbb{Z}} k} G_0(A) \otimes_{\mathbb{Z}} k \to G_0^{\mathrm{st}}(A) \otimes_{\mathbb{Z}} k \to 0.$$

Note that the rank of the map $C_A \otimes_{\mathbb{Z}} k$ is just the p-rank of C_A and $G_0(A) \otimes_{\mathbb{Z}} k$ is a vector space over k whose dimension is the number $l(A)$ of isomorphism classes of simple modules. We have

$$\mathrm{rank}_p(C_A) = l(A) - \dim(G_0^{\mathrm{st}}(A) \otimes_{\mathbb{Z}} k).$$

This establishes the equivalence of (1) and (2).

The equivalence of (1) and (3) follows from the facts that $m_0 + m_1(A) + r = l(A)$ and that stable Grothendieck groups can detect m_0 and r. □

Remark 4.2 In case of a block A of a group algebra kG over an algebraically closed field k of positive characteristic $p > 0$, the invariant factors of the Cartan matrix C_A are always powers of p. So the number of 1 in the set of invariant factors of C_A is just the p-rank of C_A and thus is the dimension of the Higman ideal, since A is symmetric.

Corollary 4.3 *Let k be an algebraically closed field. Suppose that A and B are two Frobenius k-algebras which are stably equivalent of Morita type. Then they have the same number of isomorphism classes of simple modules if and only if* $\dim(\mathrm{Im}(\theta_A)) = \dim(\mathrm{Im}(\theta_B))$ *where θ_A and θ_B are the maps introduced in Sect.* 4.

Proof This follows from Proposition 3.16 (3) and Proposition 4.1.

Corollary 4.4 *Let k be an algebraically closed field. Suppose that A and B are two finite dimensional k-algebras which are stably equivalent of Morita type. Then*

$$\dim\ (J(A) + K(A))/K(A) = \dim\ (J(B) + K(B))/K(B).$$

Proof We have

$$\begin{aligned}\dim\ A/K(A) &= \dim\ A/(J(A) + K(A)) + \dim\ (J(A) + K(A))/K(A)\\ &= l(A) + \dim\ (J(A) + K(A))/K(A).\end{aligned}$$

By the proof of Proposition 4.1, we have

$$l(A) = \mathrm{rank}_p C_A + \dim(G_0^{\mathrm{st}}(A) \otimes_{\mathbb{Z}} k)$$

and by Theorem 3.8,

$$\dim\ HH_0^{\mathrm{st}}(A) + \mathrm{rank}_p(C_A) = \dim\ HH_0(A).$$

Combining these three equalities, we obtain

$$\dim\ (J(A) + K(A))/K(A) = \dim\ HH_0^{\mathrm{st}}(A) - \dim(G_0^{\mathrm{st}}(A) \otimes_{\mathbb{Z}} k)$$

and the result follows from the fact that both $\dim HH_0^{\mathrm{st}}(A)$ and $\dim(G_0^{\mathrm{st}}(A) \otimes_{\mathbb{Z}} k)$ are invariant under stable equivalences of Morita type. □

5 Equivalent conditions of the Auslander-Reiten conjecture

Using our results in previous sections, we can now give some easily expressed equivalent conditions of the Auslander-Reiten conjecture for stable equivalences of Morita type.

Theorem 5.1 *Let k be an algebraically closed field. Let A and B be two finite dimensional k-algebras which are stably equivalent of Morita type. Then the following two statements are equivalent.*

(1) *A and B have the same number of isomorphism classes of simple modules*;
(2) $\dim HH_0(A) = \dim HH_0(B)$.
Moreover, if A and B have no semisimple summand, then (1) *and* (2) *are again equivalent to the following*
(3) *A and B have the same number of isomorphism classes of non-projective simple modules.*

Proof The equivalence of (1) and (2) follows from Theorem 3.8, Theorem 3.11 and Proposition 4.1. If A and B have no semisimple summand, then by [20], the stable equivalence of Morita type induces a bijection between the isomorphism classes of simple projective modules over A and B, and therefore (1) and (3) are equivalent in this case. □

Now we specialize to symmetric algebras. Let A be a symmetric k-algebra with a non-degenerate associative symmetric bilinear form $(\,,)$. Since $K(A)^{\perp} = Z(A)$ (cf. Sect. 2), the form $(\,,)$ induces a well-defined non-degenerate bilinear form

$$Z(A) \times A/K(A) \to k, \qquad (z, a + K(A)) \mapsto (z, a).$$

It follows that we have a duality between Hochschild homology and cohomology, that is,

$$HH_0(A) = A/K(A) \simeq \mathrm{Hom}_k(Z(A), k) = Z(A)^* = HH^0(A)^*.$$

In particular, $\dim HH_0(A) = \dim HH^0(A)$. We obtain the following

Corollary 5.2 *Let k be an algebraically closed field. Suppose that two finite dimensional k-algebras A and B are stably equivalent of Morita type and that A is symmetric. Then they have the same number of isomorphism classes of simple modules if and only if* $\dim H(A) = \dim H(B)$*, and if and only if* $\dim Z(A) = \dim Z(B)$.

Proof By [21, Corollary 2.4], a stable equivalence of Morita type preserves the property of being symmetric. Thus B is also symmetric. On the other hand, we know that $\dim H(A) = \dim Z^{\mathrm{pr}}(A) = \dim Z(A) - \dim Z^{\mathrm{st}}(A)$ and that $\dim Z^{\mathrm{st}}(A)$ is an invariant under a stable equivalence of Morita type. Now the conclusion follows from Theorem 5.1 and the remark before this corollary. □

Remark 5.3 By Theorem 5.1, for stable equivalences of Morita type, the Auslander-Reiten conjecture is equivalent to the invariance of 0-degree Hochschild homology groups. This is not true, however, for general stable equivalences. For example, let A be the path algebra over a field k given by the quiver $\overset{1}{\circ} \xrightarrow{\alpha} \overset{2}{\circ} \xrightarrow{\beta} \overset{3}{\circ}$. If we glue the source vertex 1 and the sink vertex 3 and put a zero relation in the above quiver, then we get a subalgebra B of A which is given by the following quiver

$$\begin{tikzcd} \overset{1}{\circ} \arrow[r, shift left, "\alpha"] & \overset{2}{\circ} \arrow[l, shift left, "\beta"] \end{tikzcd}$$

with relations $\alpha\beta = 0$. By [25], A and B are stably equivalent. In [14], it was shown that such stable equivalence is not of Morita type, but still induced by tensoring a pair of bimodules. On the other hand, we have $\dim HH_0(A) = 3$ and $\dim HH_0(B) = 2$. Thus $\dim HH_0(A) \neq \dim HH_0(B)$.

6 Power-p maps and the Auslander-Reiten conjecture

In this section, we work over an algebraically closed field k of positive characteristic $p > 0$ and in this special case we will give another proof of the main Theorem 5.1. As an application, we get some new computable invariants of stable equivalences of Morita type.

One of the properties of the 0-degree Hochschild homology group is that it admits a power-p map, that is,

$$\mu_p^A : A/K(A) \to A/K(A), \quad x + K(A) \mapsto x^p + K(A).$$

The above map is well-defined and additive (see, for example, [17]). We shall prove that this map restricts to the stable 0-degree Hochschild homology group. We begin with a lemma.

Lemma 6.1 *Let B be a finite dimensional k-algebra over a field k of characteristic $p > 0$. For a finitely generated projective right B-module M, the trace map*

$$tr_M : \mathrm{End}_B(M) \simeq M \otimes_B \mathrm{Hom}_B(M, B) \to B/K(B)$$

satisfies that for any $f \in \mathrm{End}_B(M)$, we have $tr(f^p) = tr(f)^p$. Consequently, the same holds for the degree zero transfer map $t_M : A/K(A) \to B/K(B)$.

Proof The first assertion should be well known, but we could not find a proof in the literature, so we include one. Suppose that $M \simeq (e_1B \oplus e_2B \oplus \cdots \oplus e_nB)$ for some primitive idempotents $e_1, \ldots, e_n \in B$. We can write an element $f \in \mathrm{End}_B(M) \simeq \mathrm{End}_B(e_1B \oplus e_2B \oplus \cdots \oplus e_nB)$ as a matrix $(f_{ij})_{1 \le i,j \le n}$ where $f_{ij} \in \mathrm{Hom}_B(e_iB, e_jB)$. The trace map gives $\mathrm{tr}_M(f) = \sum_i f_{ii}$ in $B/K(B)$. We want to prove that $\mathrm{tr}_M(f^p) = \sum_i f_{ii}^p \in B/K(B)$, therefore $\mathrm{tr}_M(f^p) = (\mathrm{tr}_M(f))^p$ in $B/K(B)$ and we are done. In fact, the trace of f^p is the sum of terms of the form

$$f_{i_1i_2}f_{i_2i_3}\cdots f_{i_{p-1}i_p}f_{i_pi_1}$$

for $1 \le i_1, \ldots, i_p \le n$. If all the indices $i_1, \ldots, i_p$ are equal to some i for $1 \le i \le n$, then the term is f_{ii}^p. We shall prove that the sum of all other terms is zero in $B/K(B)$. Consider the sequence of pairs

$$((i_1, i_2), (i_2, i_3), \ldots, (i_{p-1}, i_p), (i_pi_1))$$

which corresponds to a term above. The cyclic group of order p acts on such a sequence by permuting cyclically the pairs in this sequence. If not all of the indices are equal, then the action is free and an orbit contains p sequences. Notice that any non-trivial cyclic permutation of this sequence corresponds to a different expressed term, but these terms are all equal in $B/K(B)$. So the sum of all the terms corresponding to the cyclic permutations of a sequence is zero in $B/K(B)$ in case that not all the indices are equal. The fixed points of this action are just the sequences in which all of the indices are equal. We have now $\mathrm{tr}_M(f^p) = \sum_i f_{ii}^p$ in $B/K(B)$. The proof is complete.

The second assertion follows from the first and the construction of the transfer map. □

We now prove that the power-p map can restrict to the stable Hochschild homology group.

Corollary 6.2 $\mu_p^A(HH_0^{\mathrm{st}}(A)) \subseteq HH_0^{\mathrm{st}}(A)$.

Proof For $x \in HH_0^{\mathrm{st}}(A)$, $t_{Ae_i}(x^p) = t_{Ae_i}(x)^p = 0$ for each i.

Recall that $T_n(A) = \{x \in A \mid x^{p^n} \in K(A)\}$ is a k-subspace of A and that (see [17, (9)])

$$\bigcup_{n=0}^{\infty} T_n(A) = J(A) + K(A).$$

Lemma 6.3 *For each $n \ge 0$, $T_n(A) \subseteq HH_0^{\mathrm{st}}(A)$. As a consequence, $(J(A) + K(A))/K(A) \subseteq HH_0^{\mathrm{st}}(A)$.*

Proof Indeed, $T_n(A)/K(A) = \{x \in A/K(A) | x^{p^n} = 0\}$. So for $x \in T_n(A)/K(A)$ and for any projective A-module Ae, $t_{Ae}(x)^{p^n} = t_{Ae}(x^{p^n}) = 0$. It follows that $t_{Ae}(x) = 0$ for each Ae and $x \in HH_0^{\mathrm{st}}(A)$. □

Combining Corollary 6.2 and Lemma 6.1, we obtain the main result of this section.

Proposition 6.4 *Let k be an algebraically closed field of characteristic $p > 0$. Given an A-B bimodule ${}_AM_B$ which is finitely generated and projective as right B-module, then we have a commutative diagram*:

$$\begin{tikzcd} HH_0^{\mathrm{st}}(A) \arrow[r, "\mu_p^A"] \arrow[d, "t_M"'] & HH_0^{\mathrm{st}}(A) \arrow[d, "t_M"] \\ HH_0^{\mathrm{st}}(B) \arrow[r, "\mu_p^B"] & HH_0^{\mathrm{st}}(B). \end{tikzcd}$$

Corollary 6.5 *Let k be an algebraically closed field of characteristic $p > 0$. Let A and B be two finite dimensional k-algebras which are stably equivalent of Morita type. Then* $\dim(T_n(A)/K(A)) = \dim(T_n(B)/K(B))$.

Proof This follows from Proposition 6.4, Theorem 3.11 and the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & T_n(A)/K(A) \arrow[r] \arrow[d, dashed, "\simeq"] & HH_0^{\mathrm{st}}(A) \arrow[r, "(\mu_p^A)^{\circ n}"] \arrow[d, "t_M"', "\simeq"] & HH_0^{\mathrm{st}}(A) \arrow[d, "t_M"', "\simeq"] \\ 0 \arrow[r] & T_n(B)/K(B) \arrow[r] & HH_0^{\mathrm{st}}(B) \arrow[r, "(\mu_p^B)^{\circ n}"] & HH_0^{\mathrm{st}}(B), \end{tikzcd}$$

where $(\mu_p^A)^{\circ n}$ denotes the composition of μ_p^A with itself n times. □

Remark 6.6 Let A and B be two finite dimensional algebras over an algebraically closed field of positive characteristic. If they are derived equivalent, then $\dim(T_n(A)/K(A)) = \dim(T_n(B)/K(B))$. This fact was proved by Bessenrodt, Holm and the third author ([2]).

Now one can give an alternative proof of Theorem 1.1 in case of positive characteristic.

Corollary 6.7 *Let k be an algebraically closed field of characteristic $p > 0$. Suppose that there is a stable equivalence of Morita type between two finite dimensional k-algebras A and B. Then the following are equivalent.*

(1) *A and B have the same number of isomorphism classes of simple modules;*
(2) $\dim HH_0(A) = \dim HH_0(B)$.

Proof Since $\bigcup_{n=0}^{\infty} T_n(A)/K(A) = (J(A) + K(A))/K(A)$, by Corollary 6.5, $\dim(J(A) + K(A)/K(A))$ is invariant under a stable equivalence of Morita type. Since $l(A) = \dim(A/J(A) + K(A))$ (see [17, (5)]), we know that $l(A) = \dim(A/K(A)) - \dim(J(A) + K(A)/K(A))$. □

Now we consider symmetric algebras. Let us first recall some notations. Let k be an algebraically closed field of positive characteristic $p > 0$ and let A be a finite-dimensional symmetric k-algebra. The n-th Külshammer ideal of A is defined as the orthogonal space (with respect to the symmetrizing form on A)

$$T_n^{\perp}(A) = \{x \in A \mid (x, y) = 0 \text{ for all } y \in T_n(A)\}.$$

We then have the following fundamental lemma.

Lemma 6.8 ([[17]], No. (36) and (37)) *The subspaces $T_n^\perp(A)$ form a descending chain of ideals of the center $Z(A)$*

$$Z(A) = K(A)^\perp = T_0^\perp(A) \supseteq T_1^\perp(A) \supseteq T_2^\perp(A) \supseteq \cdots .$$

Moreover, the intersection of Külshammer ideals is the Reynolds ideal:

$$\bigcap_{i=0}^{\infty} T_n^\perp(A) = R(A) := \mathrm{Soc}(A) \cap Z(A).$$

We can now state a theorem of the third author saying that the Külshammer ideals are derived invariants. This theorem motivates the work in this article.

Theorem 6.9 ([42]) *Let k be an algebraically closed field of characteristic $p > 0$. Let A and B be symmetric k-algebras. If A and B are derived equivalent* (*that is, their derived module categories $D^b(A)$ and $D^b(B)$ are equivalent as triangulated categories*), *then there exists an algebra isomorphism $\varphi : Z(A) \to Z(B)$ such that $\varphi(T_n^\perp(A)) = T_n^\perp(B)$ for any $n \geq 0$.*

We now define a stable version of Külshammer ideals. Recall that $Z(A) \simeq \mathrm{Hom}_{A^e}(A, A)$ and the projective center $Z^{\mathrm{pr}}(A)$ is an ideal of $Z(A)$. The stable center $Z^{\mathrm{st}}(A)$ is defined to be $Z(A)/Z^{\mathrm{pr}}(A)$. Notice that by [9, Lemma 4.1 (iii)], $Z^{\mathrm{pr}}(A) = H(A) \subseteq R(A) \subseteq T_n^\perp(A)$. We define

$$T_n^{\perp,\mathrm{st}}(A) := T_n^\perp(A)/Z^{\mathrm{pr}}(A) \subseteq Z^{\mathrm{st}}(A)$$

and $R^{\mathrm{st}}(A) := R(A)/Z^{\mathrm{pr}}(A) \subseteq Z^{\mathrm{st}}(A)$. We call $T_n^{\perp,st}(A)$ the n-th stable Külshammer ideal and $R^{\mathrm{st}}(A)$ the stable Reynolds ideal, respectively. Since A is finite dimensional, when n is large, $T_n^{\perp,\mathrm{st}}(A) = R^{\mathrm{st}}(A)$.

Proposition 6.10 *Let k be an algebraically closed field of characteristic $p > 0$. Suppose that two finite dimensional k-algebras A and B are stably equivalent of Morita type and that A is symmetric. Then* $\dim T_n^{\perp,st}(A) = \dim T_n^{\perp,st}(B)$ *for any $n \geq 0$. In particular* $\dim R^{\mathrm{st}}(A) = \dim R^{\mathrm{st}}(B)$.

Proof Since $K(A)^\perp = Z(A)$, we have a well-defined non-degenerate bilinear form

$$Z(A)/T_n^\perp(A) \times T_n(A)/K(A) \to k, (\bar{z}, \bar{a}) \mapsto (z, a).$$

It follows that we have a duality between $Z(A)/T_n^\perp(A)$ and $T_n(A)/K(A)$. In particular, their dimensions are the same. Note that

$$\begin{aligned}\dim(Z(A)/T_n^\perp(A)) &= \dim(Z(A)/Z^{\mathrm{pr}}(A)) - \dim(T_n^\perp(A)/Z^{\mathrm{pr}}(A)) \\ &= \dim Z^{\mathrm{st}}(A) - \dim T_n^{\perp,\mathrm{st}}(A).\end{aligned}$$

Since $\dim(T_n(A)/K(A))$ and $Z^{\mathrm{st}}(A)$ are invariant under a stable equivalence of Morita type, so is $\dim T_n^{\perp,\mathrm{st}}(A)$. □

Remark 6.11 Notice that for symmetric algebras, the dimension of the Reynolds ideal $R(A)$ is just the number of simple modules $l(A)$. Since $\dim R(A) = \dim R^{\mathrm{st}}(A) + \dim H(A)$, the above proposition gives another proof of Corollary 5.2 in positive characteristic.

Acknowledgments The authors thank Serge Bouc for his kind explications on his transfer construction. The authors wish to express their gratitude to Steffen König for his most useful remarks and suggestions on a preliminary version of this paper. We are grateful very much to the referee whose suggestions simplified several proofs considerably.

References

1. Auslander, M., Reiten, I., Smalø, S.O.: Representation Theory of Artin Algebras. Cambridge University Press, Cambridge (1995)
2. Bessenrodt, Ch., Holm, Th., Zimmermann, A.: Generalized Reynolds ideals for non-symmetric algebras. J. Algebra **312**, 985–994 (2007)
3. Bouc, S.: Bimodules, trace généralisée, et transferts en homologie de Hochschild. http://www.lamfa.u-picardie.fr/bouc/ (1997, preprint)
4. Broué, M.: Equivalences of blocks of group algebras. In: Dlab, V., Scott, L.L. (eds.) Finite Dimensional Algebras and Related Topics, pp. 1–26. Kluwer, Dordrecht (1994)
5. Broué, M.: Higman criterion revisited. Mich. J. Math. **58**, 125–179 (2009)
6. Curtis, C.W., Reiner, I.: Methods of Representation Theory I. Wiley, New York (1981)
7. Erdmann, K.: Blocks of tame representation type and related algebras. Springer Lecture Notes in Mathematics, vol. 1428 (1980)
8. Eu, C.H., Schedler, T.: Calabi-Yau Frobenius algebras. J. Algebra **321**, 774–815 (2009)
9. Héthelyi, L., Horváth, E., Külshammer, B., Murray, J.: Central ideals and cartan invariants of symmetric algebras. J. Algebra **293**, 243–260 (2005)
10. Higman, D.G.: On orders in separable algebras. Can. J. Math. **7**, 509–515 (1955)
11. Holm, Th., Zimmermann, A.: Generalized Reynolds ideals and derived equivalences for algebras of dihedral and semidihedral type. J. Algebra **320**, 3425–3437 (2008)
12. Keller, B., Vossieck, D.: Sous les catégories dérivées. C. R. Acad. Sci. Paris **305**, 225–228 (1987)
13. Keller, B.: Invariance and localization for cyclic homology of DG algebras. J. Pure Appl. Algebra **123**, 223–273 (1998)
14. König, S., Liu, Y.M.: Gluing of idempotents, radical embeddings and two classes of stable equivalences. J. Algebra **319**, 5144–5164 (2008)
15. König, S., Liu. Y.M., Zhou. G.D.: Transfer maps in Hochschild (co)homology and applications to stable and derived invariants and to the Auslander-Reiten conjecture. Trans. Am. Math. Soc (to appear)
16. König, S., Zimmermann, A.: Derived Equivalences for Group Rings. Lecture Notes in Mathematics, vol. 1685. Springer, Berlin (1998)
17. Külshammer, B.: Group-theoretical descriptions of ring theoretical invariants of group algebras. Prog. Math. **95**, 425–441 (1991)
18. Lenzing, H.: Nilpotente Elemente in Ringen von endlicher globaler Dimension. Math. Zeit. **108**, 313–324 (1969)
19. Linckelmann, M.: Stable equivalences of Morita type for selfinjective algebras and p-groups. Math. Zeit. **223**, 87–100 (1996)
20. Liu, Y.M.: On stable equivalences induced by exact functors. Proc. Am. Math. Soc. **134**, 1605–1613 (2006)
21. Liu, Y.M.: Summands of stable equivalences of Morita type. Commun. Algebra **36**(10), 3778–3782 (2008)
22. Liu, Y.M., Xi, C.C.: Constructions of stable equivalences of Morita type for finite dimensional algebras II. Math. Zeit. **251**, 21–39 (2005)
23. Liu, Y.M., Xi, C.C.: Constructions of stable equivalences of Morita type for finite dimensional algebras I. Trans. Am. Math. Soc. **358**, 2537–2560 (2006)
24. Liu, Y.M., Xi, C.C.: Constructions of stable equivalences of Morita type for finite dimensional algebras III. J. Lond. Math. Soc. **76**(3), 567–585 (2007)
25. Martinez-Villa, R.: Algebras stably equivalent to l-hereditary algeras. In: Dlab, V., Gabriel, P. (eds.) Representation Theory II. Lecture Notes in Mathematics, vol. 832, pp. 396–431. Springer, Berlin (1980)
26. Martinez-Villa, R.: The stable equivalence for algebras of finite representation type. Commun. Algebra **13**(5), 991–1018 (1985)
27. Martinez-Villa, R.: Properties that are left invariant under stable equivalence. Commun. Algebra **18**(12), 4141–4169 (1990)
28. Martinez-Villa, R.: The stable group of a selfinjective Nakayama algebra. Commun. Algebra **19**(2), 509–517 (1991)

29. Ohtake, K., Fukushima, H.: A remark on Higman's result about separable algebras. Tsukuba J. Math. **10**, 35–46 (1986)
30. Pogorzały, Z.: Algebras stably equivalent to self-injective special biserial algebras. Commun. Algebra **22**(4), 1127–1160 (1994)
31. Pogorzały, Z.: Invariance of Hochschild cohomology algebras under stable equivalences of Morita type. J. Math. Soc. Jpn. **53**(4), 913–918 (2001)
32. Rickard, J.: Derived categories and stable equivalence. J. Pure Appl. Algebra **61**(3), 303–317 (1989)
33. Rickard, J.: Derived equivalences as derived functors. J. Lond. Math. Soc. **43**, 37–48 (1991)
34. Riedtmann, Ch.: Algebren, Darstellungsköcher, Überlagerungen und zurück. Comment. Math. Helv. **55**(2), 199–224 (1980)
35. Roggenkamp, K.W., Huber-Dyson, V.: Lattices over Orders. Lecture Notes in Mathematics, vol. 115. Springer, Berlin (1970)
36. Rouquier, R.: Derived equivalences and finite dimensional algebras. Proceedings of the International Congress of Mathematicians (Madrid, 2006), vol. II, pp. 191–221. EMS Publishing House, Zurich (2006)
37. Tang, A.P.: A note on the stable equivalence conjecture. Commun. Algebra **24**(9), 2793–2809 (1996)
38. Tachikawa, H., Wakamatsu, T.: Cartan matrices and Grothendieck groups of stable categories. J. Algebra **144**(2), 390–398 (1991)
39. Xi, C.C.: Stable equivalences of adjoint type. Forum Math. **20**, 81–97 (2008)
40. Zhou, G.D., Zimmermann, A.: Classifying tame blocks and related algebras up to stable equivalences of Morita type (2010, preprint)
41. Zhou, G.D., Zimmermann, A.: Auslander-Reiten conjecture for symmetric algebras of polynomial growth (2010, preprint)
42. Zimmermann, A.: Invariance of generalized Reynolds ideals under derived equivalences. Math. Proc. R. Ir. Acad. **107**(1), 1–9 (2007)